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Bubble Motion and Mass Transfer in Non-Newtonian Fluids:

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Part I. Single Bubble in Power Law and Bingham Fluids

The Sherwood number and drag coefficient for a single gas bubble moving in a power law fluid and a Bingham plastic fluid are obtained using perturbation methods. The perturbation parameters for power law and Bingham plastic fluids are m ($= n - 1/2$) and E ($= \tau_0 R/U\mu_0$), respectively. It is found that in the case of power law fluid, mass transfer and drag increase with increasing pseudoplasticity. These theoretical results are found to be in good agreement with the available experimental data and the data obtained in the present study. In the case of Bingham plastic fluid, mass transfer and drag are found to increase with increase in the Bingham number N_B ($= 2\epsilon$). Contours of plug flow regions, where local stresses are less than the yield stress, are obtained as a function of the Bingham number N_B. These results qualitatively predict the zero terminal velocity observed for bubble motion in liquids with very high yield stress. They are also in good agreement with the trends of the results obtained previously for solid sphere motion in Bingham plastic fluids.

SCOPE

There are numerous examples of bubble motion and mass transfer in non-Newtonian fluids in antibiotic fermentations, biological wastewater treatment, polymer processes, and food processing (Blanch and Bhavaraju, 1976; Astarita and Mashelkar, 1977). In order to understand these complex process situations, a first approach is presented in this study of single bubble motion and mass transfer in non-Newtonian fluids. The case of multiple bubbles is treated in a following paper. In general, polymer melts, fermentation broths, and some waste treatment processes exhibit pseudoplastic or Bingham plastic behavior. Hence, the effects of pseudoplasticity and yield stress are investigated in the present study.

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Hirose and Moo-Young (1969) obtained the drag and mass transfer coefficients for a single bubble moving in a power law fluid and predicted an enhancement in mass transfer and drag with increasing pseudoplasticity. Experiments supporting this enhancement were, however, found to be higher than the theoretical predictions. Wellek and Huang (1970) examined mass transfer enhancement factor by solving the pertinent equations numerically, using the stream functions obtained by Nakano and Tien (1968) for Newtonian liquid drops moving in a power law continuous phase. Their results were found to deviate significantly from the results of Hirose and Moo-Young (1969) for large values of the Peclet number. The predicted values of the Sherwood number were 10% lower than the predictions of Hirose and Moo-Young (1969). Later, Moo-Young and Hirose (1972) showed that the Sherwood number obtained by them, using the thin boundary-layer approximation, is the minimum, and hence the numerical calculations of Wellek and Huang (1970) are in error at high Peclet numbers. Gurkan and Wellek (1976) presented a corrected and modified version of Wellek and Huang's (1970) results using Mohan's (1974) stream functions instead of Nakano and Tien's (1968) stream functions.

No work has been reported in literature on bubble motion and mass transfer in Bingham plastic fluids. Astarita and Apuzzo (1965) found that the terminal velocity of small bubbles approaches zero as the yield stress increases. Considerable experimental work and some theoretical work have been reported on solid sphere motion in Bingham plastic fluids (Adachi, 1973).

The objectives of the present work are thus to examine drag and mass transfer coefficients for a single bubble moving in a power law fluid and Bingham plastic fluid, in the creeping flow regime, and to study the effects of the pseudoplastic index n and the yield stress τ_o on the bubble motion and mass transfer.

CONCLUSIONS AND SIGNIFICANCE

Continuous phase Sherwood number and drag coefficient for creeping motion of a single bubble in a power law fluid are obtained using perturbation methods. Results show that the Sherwood number and the drag coefficient increase with increasing pseudoplasticity. The enhancement in mass transfer predicted in this study shows a good agreement with the experimental data of Hirose (1970). The enhancement of drag coefficient predicted theoretically is also in good agreement with the experimental data obtained in this study and the data of Acharya et al. (1977).

For the case of creeping motion of a single bubble in a Bingham plastic fluid, the Sherwood number and drag coefficient are found to increase with an increase in the Bingham number $N_B = (2\tau_o R/U\mu_o)$. The present prediction, that the plug flow regions approach the bubble surface as the yield stress is increased, is consistent with the observations of the present work and of Astarita and Apuzzo (1965), that small bubbles show less motion in fluids with high yield stress. The trends of the results are also in good agreement with the trends of the results of Adachi (1973) for the case of a solid sphere motion.

Stream functions for single Newtonian drop motion in a power law fluid were first obtained by Nakano and Tien (1968), from which equations for bubble motion can be obtained as a limiting case. Hirose and Moo-Young (1969) obtained an approximate analytical solution of the equations of motion and obtained relations for mass transfer and drag for a single bubble moving in a power law fluid. Their theoretical results underpredicted the experimental mass transfer and drag coefficient data. Wellek and Huang (1970) calculated the mass transfer coefficients for Newtonian drop motion in power law fluid, using the stream functions obtained by Nakano and Tien (1968). Their results for large values of Peclet number are found to deviate significantly from the results of Hirose and Moo-Young (1969).

No work was reported in the literature on the mass transfer and drag coefficients for a gas bubble moving in a Bingham plastic fluid. Adachi (1973) obtained the drag coefficient for a solid sphere moving in a Bingham plastic fluid, using variational methods, and Koizumi (1974) obtained the drag coefficient using perturbation methods. This paper presents a solution for the drag and mass transfer coefficients for single bubble motion in power law fluid and Bingham plastic fluid, obtained by perturbation methods.

Consider the steady, incompressible, creeping flow of a power law fluid around a spherical gas bubble of radius R. Let the fluid be moving with a superficial velocity U in the positive Z direction as shown in Figure 1. Since the flow is axisymmetric, it is independent of the angle ϕ. The equations of continuity and motion for axisymmetric creeping flow can be written as follows:

$$\frac{1}{\tilde{r}^2}\frac{\partial}{\partial \tilde{r}}(\tilde{r}^2\tilde{v}_r) + \frac{1}{\tilde{r}\sin\theta}\frac{\partial}{\partial\theta}(\tilde{v}_\theta\sin\theta) = 0 \quad (1)$$

Motion

$$\frac{\partial\tilde{p}}{\partial\tilde{r}} = \frac{1}{\tilde{r}^2}\frac{\partial}{\partial\tilde{r}}(\tilde{r}^2\tilde{\tau}_{rr}) + \frac{1}{\tilde{r}\sin\theta}\frac{\partial}{\partial\theta}(\tilde{\tau}_{r\theta}\sin\theta) - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{\tilde{r}} \quad (2)$$

$$\frac{\partial\tilde{p}}{\partial\theta} = \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}(\tilde{r}^2\tilde{\tau}_{r\theta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{\tau}_{\theta\theta}\sin\theta) + \tilde{\tau}_{r\theta} - \tilde{\tau}_{\phi\phi}\cot\theta \quad (3)$$

where $\tilde{p}$ is the isotropic pressure and $\tilde{\tau}_{r\theta}$, $\tilde{\tau}_{\theta\theta}$, $\tilde{\tau}_{rr}$, and $\tilde{\tau}_{\phi\phi}$ are the components of stress tensor in the spherical polar

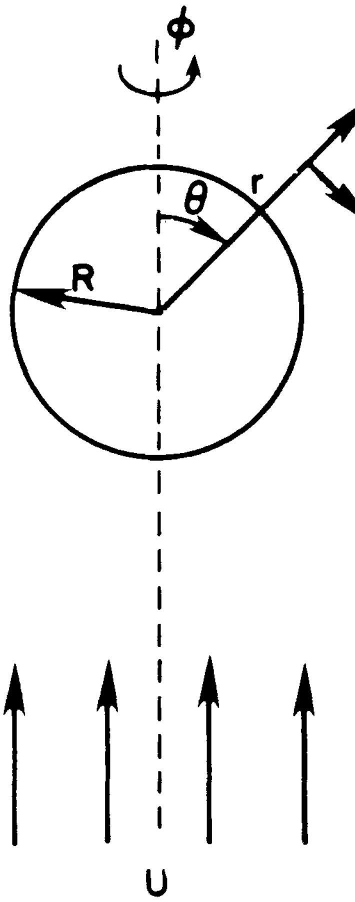

Fig. 1. Coordinate system for single bubble motion.

coordinate system (r, θ, ϕ). The constitutive equation for both the power law model and the Bingham plastic model can be written as

$$\underline{\underline{\tilde{\tau}}} = 2\tilde{\eta}\,\underline{\underline{\tilde{d}}} \tag{4}$$

where $\tilde{\eta}$ is given the power-law model

$$\tilde{\eta} = K\tilde{J}^{2m} \tag{5}$$

where

$$m = \frac{n-1}{2} < 0$$

n is the power law index, and K is the consistency index. Bingham plastic model:

$$\tilde{\eta} = \mu_o + \frac{\tau_o}{\tilde{J}} \quad \text{for} \quad ½\,\tilde{\Pi}_\tau > \tau_o^2$$

and

$$\underline{\underline{\tilde{d}}} = 0 \quad \text{for} \quad ½\,\tilde{\Pi}_\tau < \tau_o^2 \tag{6}$$

In both these models $\tilde{J}$ is given by

$$\tilde{J} = \sqrt{2\,\tilde{\Pi}_d} \tag{7}$$

where μ_o is the plastic viscosity, τ_o is the yield stress, and $\tilde{\Pi}_d$ and $\tilde{\Pi}_\tau$ are the second invariants of the rate of deformation tensor and the stress tensor, respectively. The later two quantities can be expressed as

$$\tilde{\Pi}_d = \tilde{d}^2_{rr} + \tilde{d}^2_{\theta\theta} + \tilde{d}^2_{\phi\phi} + 2\,\tilde{d}^2_{r\theta}$$
$$\tilde{\Pi}_\tau = \tilde{\tau}^2_{rr} + \tilde{\tau}^2_{\theta\theta} + \tilde{\tau}^2_{\phi\phi} + 2\,\tilde{\tau}^2_{r\theta} \tag{8}$$

The constitutive equations and the equations of continuity and motion can be nondimensionalized as

$$r = \tilde{r}/R \qquad v_i = \tilde{v}_i/U$$
$$d_{ij} = \tilde{d}_{ij} \Big/ \frac{U}{R} \qquad J = \tilde{J} \Big/ \frac{U}{R}$$
$$\tau_{ij}\big|_{\text{power law}} = \tilde{\tau}_{ij}/K\left(\frac{U}{R}\right)^n$$
$$\tau_{ij}\big|_{\text{Bingham}} = \tilde{\tau}_{ij}/\mu_o\frac{U}{R} \tag{9}$$
$$\eta\big|_{\text{power law}} = \tilde{\eta}/K\left(\frac{U}{R}\right)^{n-1}$$
$$\eta\big|_{\text{Bingham}} = \tilde{\eta}/\mu_o$$

Thus obtaining:

Equation of continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) = 0 \tag{10}$$

Equations of motion

$$\frac{\partial p}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\tau_{r\theta}\sin\theta) - \left(\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right) \tag{11}$$

$$\frac{\partial p}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tau_{\theta\theta}\sin\theta) + \tau_{r\theta} - \tau_{\theta\theta}\cot\theta \tag{12}$$

Constitutive equations

Power-law

$$\underline{\underline{\tau}} = 2\eta\underline{\underline{d}} \tag{13}$$

where

$$\eta = J^{2m} = (2\Pi_d)^m = \gamma^m \tag{14}$$

Bingham

$$\underline{\underline{\tau}} = 2\eta\underline{\underline{d}} \quad \text{for} \quad ½\,\Pi_\tau > \epsilon^2 \tag{15}$$

$$\underline{\underline{d}} = 0 \quad \text{for} \quad ½\,\Pi_\tau < \epsilon^2 \tag{16}$$

where

$$\eta = 1 + \epsilon/J \tag{17}$$

$$J = \sqrt{2\Pi_d} = \gamma^{½} \tag{18}$$

$$\epsilon = \frac{\tau_o R}{U\mu_o} = ½\,N_B \tag{19}$$

Now, the equations of motion (11) and (12) can be solved using the constitutive Equations (13) and (15).

POWER-LAW FLUID

Expanding the stream function, velocity, and deformation gradient in the form of a power series in the perturbation parameter m, we obtain

$$\psi = \psi_0 + m\psi_1 + m^2\psi_2 + \cdots\cdots \tag{20}$$

$$v_i = v_{i0} + m\,v_{i1} + m^2\,v_{i2} + \cdots\cdots \tag{21}$$

$$d_{ij} = d_{ij0} + m\,d_{ij1} + m^2\,d_{ij2} + \cdots\cdots \tag{22}$$

Similarly, γ can be expressed as

$$\gamma = \gamma_0 + m\gamma_1 + m^2\gamma_2 + \cdots \quad (23)$$

Hence

$$\tau_{ij} = 2\gamma^m d_{ij}$$
$$= 2\gamma^m [d_{ij0} + m\, d_{ij1} + 0(m^2)] \quad (24)$$

For reasons which will be clear later, let

$$\gamma_0 = \sigma_2 (1 + H) \quad (25)$$

where σ is a constant, and H is a function of r and θ such that $H^2 \leqq 1$. Then

$$\gamma_0{}^m = \sigma^{2m}[1 + m(H - H^2/2 + H^3/3 - \cdots) + 0(m^2)] \quad (26)$$

Hence τ_{ij} can be expressed as

$$\tau_{ij} = 2\sigma^{2m}[d_{ij0} + m(d_{ij1} + Z\, d_{ij0})] \quad (27)$$

where

$$Z = H - H^2/2 + H^3/3 \quad (28)$$

Here, the fourth and higher powers of H are neglected since $H^2 < 1$. Substituting this Equation (27) in the equations of motion (11) and (12) and rearranging in terms of the stream function by eliminating the isotropic pressure, we obtain

$$\frac{1}{\sin\theta}\nabla^4\psi_o + m\left[\frac{1}{\sin\theta}\nabla^4{}_1\psi_1 - 2\{\nabla_1(Z\, d_{r\theta 0}) + \nabla_2(Z\, d_{rr0}) - \nabla_3(Z\, d_{\theta\theta 0})\}\right] = 0 \quad (29)$$

where

$$\nabla^4 = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2$$

$$\nabla_1 = \left[\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - r\frac{\partial^2}{\partial r^2} - 4\frac{\partial}{\partial r} + \frac{\cot\theta}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin^2\theta}\right]$$

$$\nabla_2 = \left[\frac{\partial^2}{\partial r\partial\theta} + \frac{3}{r}\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial r}\right]$$

$$\nabla_3 = \left[\frac{\partial^2}{\partial r\partial\theta} + 2\cot\theta\frac{\partial}{\partial r}\right]$$

The relevent boundary conditions are

(1) $v_r = 0$
at $r = 1$
(2) $\tau_{r\theta} = 0$

(3) $v_r = \cos\theta$
as $r \to \infty$ (30)
(4) $v_\theta = -\sin\theta$

Zero-Order Solution

The zero-order solution is obtained by solving the following equation with the above boundary conditions:

$$\nabla^4\psi_o = 0 \quad (31)$$

The solution of this equation is (Levich, 1962)

$$\psi_o = f_o(r)\sin^2\theta$$
$$= \tfrac{1}{2}(r - r^2)\sin^2\theta \quad (32)$$

Hence, from Equation (14)

$$\gamma_0 = 3\cos^2\theta/r^4 \quad (33)$$

Comparing with Equation (25), we obtain

$$\sigma = \sqrt{3} \quad (34)$$

and

$$H = \cos^2\theta/r^4 - 1 \quad (35)$$

Thus, $H^2 \leqq 1$ for $r = 1$, where equality holds when $r = 1$ and $\theta = \Pi/2, 3\Pi/2$ or when $r \to \infty$. Hence, this assumption is valid in the region of interest, except near the equator.

First-Order Solution

By equating equal powers of m in Equation (29), the following differential equation is to be solved for the first-order approximation:

$$\frac{1}{\sin\theta}\nabla^4\psi_1 = 2\{\nabla_1(d_{r\theta 0}Z) + \nabla_2(d_{rr0}Z) - \nabla_3(d_{\theta\theta 0}Z)\} \quad (36)$$

Substituting for Z and H from Equations (28) and (35) and for d_{rr0}, $d_{r\theta 0}$, and $d_{\theta\theta 0}$, we get

$$\nabla^4\psi_1 = S_1\sin^2\theta + S_2\sin^4\theta + S_3\sin^6\theta + S_4\sin^8\theta \quad (37)$$

where

$$S_1 = 108\,r^{-7} - 180\,r^{-11} + 84\,r^{-15}$$
$$S_2 = -108\,r^{-7} + 360\,r^{-11} - 252\,r^{-15}$$
$$S_3 = -180\,r^{-11} + 252\,r^{-15}$$
$$S_4 = 84\,r^{-15}$$

Assuming the solution of the Equation (37) of the form

$$\psi_i = \phi_1\sin^2\theta + \phi_2\sin^4\theta + \phi_3\sin^6\theta + \phi_4\sin^8\theta \quad (38)$$

and solving for ϕ_1, ϕ_2, ϕ_3, and ϕ_4 using the boundary conditions (30), we obtain

$$\psi = \psi_0 + m\psi_1$$
$$= f_0\sin^2\theta + m(\phi_1\sin^2\theta + \phi_2\sin^4\theta + \phi_3\sin^6\theta + \phi_4\sin^8\theta) \quad (39)$$

where

$$f_0 = 0.5\,(r - r^2)$$
$$\phi_4 = 0.026\,r^{-5} - 0.035\,r^{-7} + 0.0087\,r^{-11}$$
$$\phi_3 = -0.053\,r^{-3} + 0.073\,r^{-5} - 0.014\,r^{-7} - 0.0064\,r^{-11}$$
$$\phi_2 = 0.0586\,r^{-1} - 0.159\,r^{-3} - 0.140\,r^{-5} + 0.25\,r^{-7} - 0.01\,r^{-11} + 0.86\,r^{-3}\ln r$$
$$\phi_1 = 1.1\,r - 1.3\,r^{-1} + 0.24\,r^{-3} + 0.04\,r^{-5} - 0.09\,r^{-7} + 0.01\,r^{-11} - 0.69\,r^{-3}\ln r$$

Mass Transfer

The mass transfer relation for a single bubble moving with a mobile interface in a liquid can be obtained by using the following equation derived by Baird and Hemielec (1962):

$$N_{Sh} = \sqrt{\frac{2}{\pi}}\left[\int_0^\pi \tilde{v}_\theta\big|_{r=R}\sin^2\theta\, d\theta\right]^{1/2} N_{Pe}{}^{1/2} \quad (40)$$

For a single bubble moving in power law fluid, substituting $\tilde{v}_\theta|_{r=R}$ obtained from Equation (39) in the above Equation (40) and integrating, we obtain

$$N_{Sh} = 0.65\, F^M(m)\, N_{Pe}{}^{1/2} \quad (41)$$

where

$$F^M(m) = (1 - 3.24\,m)^{1/2} \quad (42)$$

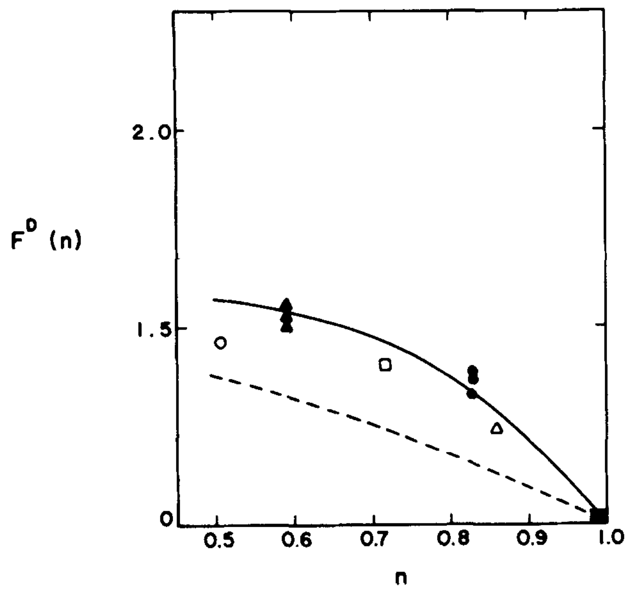

Fig. 2. Correction factor for drag coefficient [$F^D(n)$] as a function of the power law index.

- - - - - - - - Hirose and Moo-Young (1969)
———— present work
○ 1.00% CMC
□ 0.75% CMC
Acharya et al. (1977)
△ 0.50% CMC
■ 0.05% CMC
● 0.10% carbopol
present work
▲ 0.15% carbopol

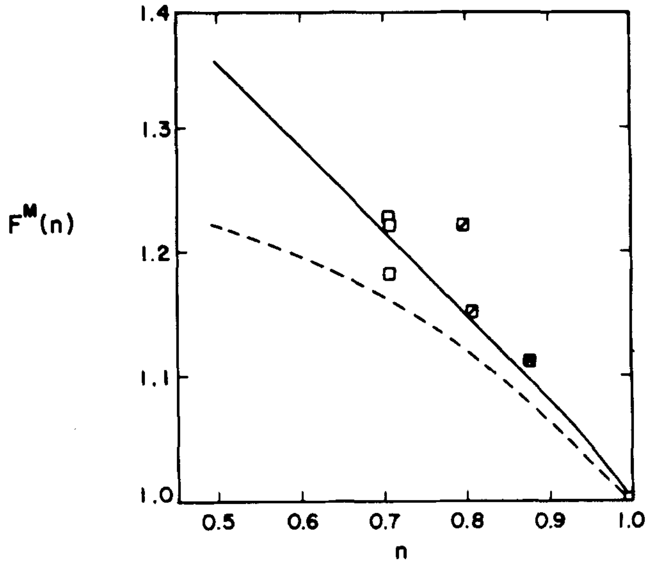

Fig. 3. Correction factor for mass transfer coefficient [$F^M(n)$] as a function of the power law index.

- - - - - - - - Hirose and Moo-Young (1969).
———— present work
⊠ 2.5% CMC
⧄ 3.0% CMC
□ 1.14% HEC
Hirose (1970)

Figure 2 shows this theoretical prediction along with the theoretical and experimental results of Hirose and Moo-Young (1969).

Drag Force

Let the isotropic pressure be expressed as

$$p = \gamma_0^m (p_0 + m\, p_1) \tag{43}$$

Substituting for γ_0^m and simplifying, we get

$$p = \sigma^{2m}(p_0 + m\, P_1) \tag{44}$$

where

$$P_1 = p_0 Z + p_1 \tag{45}$$

Substituting Equations (44) and (27) for p and τ_{ij} in Equation (12), collecting equal powers of m, and integrating, we get

$$\tilde{P_1}|_{\tilde{r} = R} = 24.9 \cos\theta - 9.2 \cos^3\theta + 2.1 \cos^5\theta - 1.1 \cos^7\theta \tag{46}$$

Substituting for d_{rr0} and d_{rr1} in Equation (27), we get

$$\tilde{\tau}_{rr}|_{\tilde{r} = R} = -7.6 \cos\theta + 9.1 \cos^3\theta - 3.1 \cos^5\theta + 1.5 \cos^7\theta \tag{47}$$

Let the dimensionless drag force F_D be defined as

$$F_D = \frac{F}{2\pi R^2 \mu_0 \left(\frac{U}{R}\right)^{2m+1}} = - \int_0^\pi (\tilde{p} - \tilde{\tau}_{rr})|_{\tilde{r} = R} \sin\theta \cos\theta \, d\theta \tag{48}$$

so that substituting for $\tilde{p}|_{\tilde{r} = R}$ and $\tilde{\tau}_{rr}|_{\tilde{r} = R}$ and integrating, we obtain

$$F_D = 3^m(2 - 15.3\, m) \tag{49}$$

We define the drag coefficient as

$$C_D = \frac{F}{\frac{1}{2}\pi R^2 \rho U^2} = \frac{16\, F^D(m)}{N_{Re'}} \tag{50}$$

Thus

$$F^D(m) = 3^m\, 2^m (1 - 7.66\, m) \tag{51}$$

Figure 3 shows $F^D(m)$ as a function of n, along with the theoretical prediction of Hirose and Moo-Young (1969) and the experimental results of Acharya et al. (1977) and the present work.

BINGHAM PLASTIC FLUID

Expanding the stream function, velocity, and deformation gradient in the form of a power series in the perturbation parameter ϵ, we obtain

$$\psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \text{-------} \tag{52}$$

$$v_i = v_{i0} + \epsilon v_{i1} + \epsilon^2 v_{i2} + \text{------} \tag{53}$$

$$d_{ij} = d_{ij0} + \epsilon d_{ij1} + \epsilon^2 d_{ij2} + \text{----} \tag{54}$$

Similarly

$$J = [2\Pi_d]^{1/2} = 2(\Pi_{d0} + \epsilon\Pi_{d1} + \epsilon^2\Pi_{d2} + \text{--})^{1/2}$$

$$J_0 = \left[1 + \epsilon\frac{\gamma_1}{\gamma_0} + \epsilon_2\frac{\gamma_2}{\gamma_0} \cdots \right]^{1/2} \tag{55}$$

where

$$J_0 = [2\Pi_{d0}]^{1/2} \tag{56}$$

Hence

$$\begin{aligned} \tau_{ij} &= 2\eta d_{ij} \\ &= 2(1 + \epsilon/J)d_{ij} \\ &= 2\left[1 + \frac{\epsilon}{J_0}\left(1 + \epsilon\frac{\gamma_1}{\gamma_0} + \epsilon^2\frac{\gamma_2}{\gamma_0}\right)^{-1/2} \right] d_{ij} \end{aligned}$$

$$= 2\left[1 + \frac{\epsilon}{J_0} + 0(\epsilon^2)\right] d_{ij} \quad (57)$$

Substituting for d_{ij} from Equation (54), we get

$$\tau_{ij} = 2\left[d_{ij0} + \epsilon\left(\frac{d_{ij0}}{J_0} + d_{ij1}\right) + 0(\epsilon^2)\right] \quad (58)$$

Eliminating the isotropic pressure p from Equations (11) and (12) and substituting Equations (58) and rewriting in terms of the stream function, we get

$$\frac{1}{\sin}\nabla^4\psi_0 + \epsilon\left[\frac{1}{\sin\theta}\nabla^4\psi_1 - 2\left\{\left(\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - r\frac{\partial^2}{\partial r^2}\right.\right.\right.$$

$$\left. - 4\frac{\partial}{\partial r} + \frac{\cot\theta}{r}\frac{\partial}{\partial\theta} - \frac{1}{r\sin^2\theta}\right)\frac{d_{r\theta 0}}{J_0}$$

$$+\left(\frac{\partial^2}{\partial r\partial\theta} + \frac{3}{r}\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial r}\right)\frac{d_{rr0}}{J_0}$$

$$\left.\left. - \left(\frac{\partial^2}{\partial r\partial\theta} + 2\cot\theta\frac{\partial}{\partial\theta}\right)\frac{d_{\theta\theta 0}}{J_0}\right\}\right] + 0(\epsilon^2) = 0 \quad (59)$$

The relevant boundary conditions are the same as given in Equation (30).

Zero-Order Solution

Since the zero-order solution corresponds to the flow of a Newtonian fluid over a gas bubble, the result will be the same as that obtained in the previous section. Hence, ψ_0, γ_0, σ, and H are the same as those given before.

First-Order Solution

Equating equal powers of ϵ in Equation (59), we obtain the following differential equation to be solved for the first-order approximation:

$$\frac{1}{\sin\theta}\nabla^4\psi_1 = 2\left[\frac{1}{r}\frac{\partial^2}{\partial\theta^2} - r\frac{\partial^2}{\partial r^2} - 4\frac{\partial}{\partial r} + \frac{\cot\theta\partial}{r\,\partial\theta}\right.$$

$$\left. - \frac{1}{r\sin^2\theta}\right]\frac{d_{r\theta 0}}{J_0} + 2\left[\frac{3}{2}\frac{\partial^2}{\partial r\partial\theta} + \frac{3}{r}\frac{\partial}{\partial\theta}\right]\frac{d_{rr0}}{J_0} \quad (60)$$

Since

$$J_0 = \gamma_0^{1/2} = \sqrt{3}(1+H)^{1/2} \quad (61)$$

$$\frac{1}{J_0} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{2}H + \frac{3}{8}H^2 - \frac{5}{16}H^3\right) \quad (62)$$

where fourth and higher powers of H are neglected since $H^2 < 1$. Substituting for d_{rr0}, $d_{r\theta 0}$, H, and $1/J_0$ in Equation (60) and simplifying, we obtain

$$\nabla^4\psi_1 = S_1\sin^2\theta + S_2\sin^4\theta + S_3\sin^6\theta + S_4\sin^8\theta \quad (63)$$

where

$$S_1 = -45.47\,r^{-7} + 90.93\,r^{-11} - 45.47\,r^{-15}$$

$$S_2 = 45.47\,r^{-7} - 181.86\,r^{-11} + 136.40\,r^{-15}$$

$$S_3 = 90.93\,r^{-11} - 136.40\,r^{-15}$$

$$S_4 = -45.47\,r^{-15}$$

Assuming the solution of the form

$$\psi_1 = \phi_1\sin^2\theta + \phi_2\sin^4\theta + \phi_3\sin^6\theta + \phi_4\sin^8\theta \quad (64)$$

and solving for ϕ_1, ϕ_2, ϕ_3, and ϕ_4 using the above boundary conditions, we get

$$\psi \doteq \psi_0 + \epsilon\psi_1$$
$$= f_0\sin^2\theta + \epsilon[\phi_1\sin^2\theta + \phi_2\sin^4\theta + \phi_3\sin^6\theta + \phi_4\sin^8\theta] \quad (65)$$

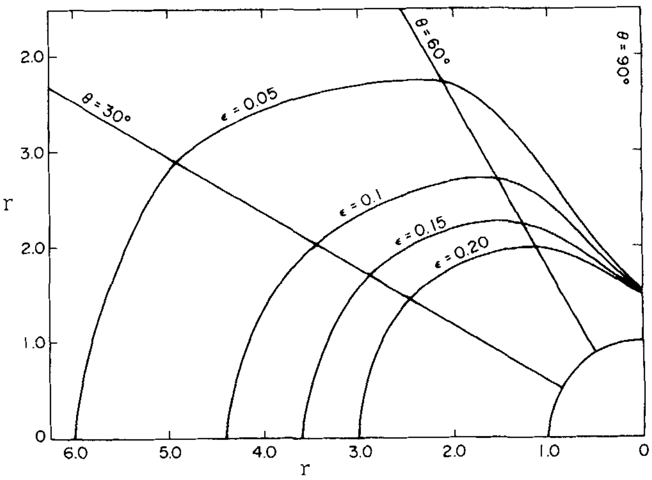

Fig. 4. Contours of plug flow regions as a function of ϵ.

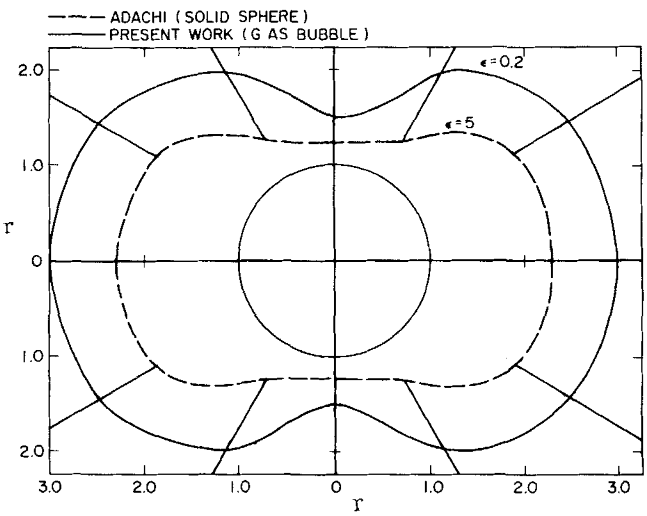

Fig. 5. Contours of plug flow regions for a gas bubble (ϵ = 0.2) and solid sphere (ϵ = 5.0).

where

$$f_0 = 0.5\,(r - r^2)$$

$$\phi_4 = -0.014\,r^{-5} + 0.019\,r^{-7} - 0.005\,r^{-11}$$

$$\phi_3 = 0.007\,r^{-3} - 0.026\,r^{-5} + 0.024\,r^{-7} - 0.005\,r^{-11}$$

$$\phi_2 = 0.24\,r^{-1} - 0.125\,r^{-3} + 0.055\,r^{-5} - 0.077\,r^{-7} + 0.008\,r^{-11} - 0.361\,r^{-3}\ln r$$

$$\phi_1 = -0.082\,r + 0.006\,r^{-1} + 0.062\,r^{-3} - 0.018\,r^{-5} + 0.035\,r^{-7} - 0.003\,r^{-11} + 0.289\,r^{-3}\ln r$$

This solution is valid in the region where $\frac{1}{2}\Pi_\tau > \epsilon^2$, and plug flow regions exist where $\frac{1}{2}\Pi_\tau < \epsilon^2$. In order to find the contours which separate the plug flow regions from regions where velocity gradients exist, $\frac{1}{2}\Pi_\tau$ was calculated using Equation (58) for various values of ϵ. These contours are shown in Figure 4, from which it is clear that the plug flow regions approach the bubble as ϵ increases. One such contour for ϵ = 0.2 was shown in Figure 5 along with the contour for ϵ = 5.0 obtained by Adachi (1973) for the case of Bingham fluid flow over a solid sphere.

Mass Transfer

Mass transfer relation for a single gas bubble with a mobile interface moving in a Bingham fluid is obtained

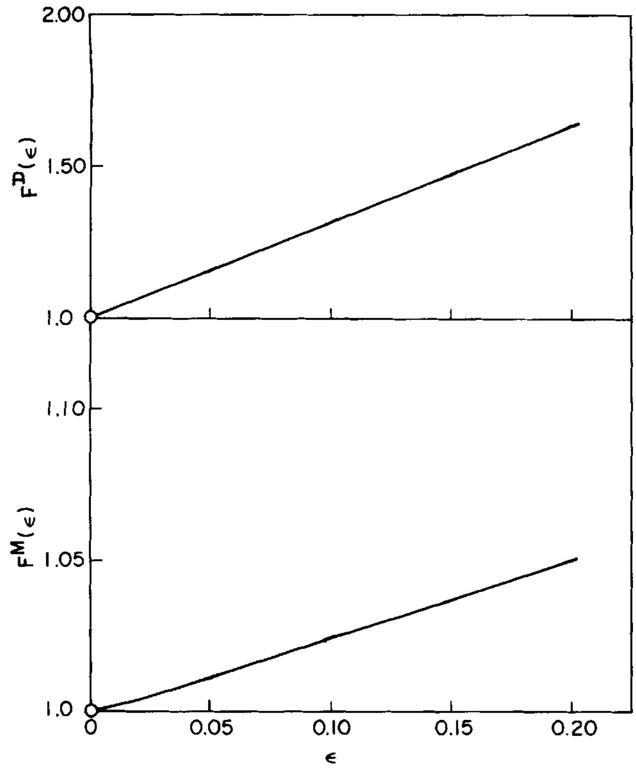

Fig. 6. Correlation factors for mass transfer and drag coefficients as a function of ϵ.

by calculating $\tilde{v}_\theta|_{r=R}$ from Equation (65) and substituting in Equation (40):

$$N_{Sh} = 0.65\, F^M(\epsilon)\, N_{Pe}{}^{1/2} \qquad (66)$$

where

$$F^M(\epsilon) = (1 + 0.5\epsilon)^{1/2} \qquad (67)$$

Figure 6 shows this theoretical relation $F^M(\epsilon)$ as function of ϵ.

Drag Force

Let the isotropic pressure be expressed as

$$p = p_0 + \epsilon\, p_1 \qquad (68)$$

Substituting Equations (68) and (58) for p and τ_{ij} in Equation (12), collecting equal powers of ϵ, and integrating, we obtain

$$\tilde{p}_1|_{\tilde{r}=R} = -8.96 \cos\theta + 3.93 \cos^3\theta - 2.44 \cos^5\theta + 0.52 \cos^7\theta \qquad (69)$$

Substituting for d_{rr0} and d_{rr1} in Equation (58), we obtain

$$\tilde{\tau}_{rr}|_{\tilde{r}=R} = 3.54 \cos\theta - 3.40 \cos^3\theta + 1.95 \cos^5\theta - 0.49 \cos^7\theta \qquad (70)$$

Substituting for $p|_{\tilde{r}=R}$ and $\tau_{rr}|_{r=R}$ in Equation (48) and integrating, we obtain

$$F_D = 2 + 6.44\epsilon \qquad (71)$$

Defining the drag coefficient as

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 \pi R^2} = \frac{16\, F^D(\epsilon)}{N_{Re'}} \qquad (72)$$

we obtain

$$F^D(\epsilon) = 1 + 3.22\epsilon \qquad (73)$$

Figure 6 shows $F^D(\epsilon)$ as a function of ϵ.

DISCUSSION

Correction factors for mass transfer and drag coefficients for a single bubble moving in a power law fluid obtained in the present analysis were found to be in good agreement with the experimental data. The deviation of the experimental results of mass transfer (Hirose, 1970) from the present work is within 0.5%, and from Hirose and Moo-Young (1969) prediction is within 5%. Experimental results of drag coefficients of the present work and that of Acharya et al. (1977) are in good agreement within 4% of the present theoretical predictions and within 12% of the Hirose and Moo-Young (1969) prediction.

In the case of gas bubbles moving in power law fluids with an immobile interface, the drag coefficients can be obtained from the results of Tomita (1959), Wasserman and Slattery (1964), and Nakano and Tien (1968) and the corresponding mass transfer coefficients obtained from the results of Wellek and Huang (1970) and Astarita and Mashelkar (1977).

In the case of single bubble motion in Bingham plastic fluids, the trends of the results are in good agreement with the trends of the previous results on the motion of solid sphere (Adachi, 1973). The present prediction that the plug flow regions approach the gas bubble, as the yield stress is increased is consistent with the observation that small bubbles remain motionless in fluids with high yield stress (Astarita and Apuzzo, 1965).

ACKNOWLEDGMENT

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NOTATION

C_D = drag coefficient
$\underline{\underline{d}}$ = rate of deformation tensor
$F^M(m)$ = correction factor for mass transfer coefficient defined by Equation (42)
$F^D(m)$ = correction factor for drag coefficient defined by Equation (51)
$F^M(\epsilon)$ = correction factor for mass transfer coefficient defined by Equation (67)
$F^D(\epsilon)$ = correction factor for drag coefficient defined by Equation (73)
H = function defined by Equation (35)
J = function defined by Equation (7)
m = $(n - 1)/2$
n = power law index
N_B = $2\tau_0 R/U\mu_0$, Bingham number
N_{Pe} = $2RU/\mathfrak{D}$, Peclet number
$N_{Re'}$ = $\rho U^{2-n}(2R)^n/\mu_0$, Reynolds number
N_{Sh} = $2R\, k_L/\mathfrak{D}$, Sherwood number
p = isotropic pressure
$\tilde{r}$ = radial distance from the center of the bubble
R = bubble radius
U = free rise velocity
v_i = velocity component
Z = function defined by Equation (28)

Greek Letters

γ = function defined by Equation (18)
ϵ = $N_B/2$
η = apparent viscosity, defined by Equations (5) and (6)
μ_o = parameter defined by Equations (5) and (6)
Π_d = second invariant of the deformation tensor
Π_τ = second invariant of the stress tensor

ρ = density of continuous phase
σ = constant defined by Equation (25)
$\underline{\underline{\tau}}$ = stress tensor
ψ = stream function

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Part II. Swarm of Bubbles in a Power Law Fluid

The Sherwood number and the drag coefficient for a swarm of bubbles moving in a power law fluid are obtained by an approximate solution of equations of motion in the creeping flow regime. The effect of gas holdup on the motion and mass transfer of a swarm of bubbles with a mobile interface moving in a power law fluid was obtained using Happel's free surface cell model. Since the presence of surfactants may cause small bubbles to behave as solid spheres, expressions for Sherwood number and drag coefficient are also obtained by using the results of power-law fluid motion over assemblages of solid spheres. It is predicted that, unlike the case of a single bubble, the drag coefficient correction factor and the mass transfer correction factor decrease with increasing pseudoplasticity.

SCOPE

Mass transfer in gas dispersions in non-Newtonian fluids is important in wide range of chemical processes, for example, fermentation, sewage treatment, direct contact blood oxygenation, removal of monomers from polymers during the finishing stages, etc. An adequate understanding of the problem of single bubble motion in non-Newtonian fluids together with pertinent experimental data has been available (see Part I). However, information on multiple bubble motion and mass transfer in non-Newtonian fluids is not available. The hydrodynamic field around a bubble is affected significantly by the presence of other bubbles. The purpose of this work is to study the effect of gas holdup and pseudoplasticity on the motion and mass transfer of multiple bubbles under creeping flow conditions.

CONCLUSIONS AND SIGNIFICANCE

The rise velocity of a swarm of bubbles with a mobile interface, moving in a power law fluid, is found to decrease with increasing gas holdup, as found in the case of Newtonian continuous phase. The ratio of swarm velocity to single-bubble velocity for $n = 1$ reduces to the results obtained by Gal-Or and Waslo (1968). Unlike the results for single-bubble motion, increasing pseudoplasticity is found to increase the swarm velocity. The same results were obtained for gas bubbles with immobilized interfaces moving in power law fluids. The effect of pseudoplasticity found in the present work is in agreement with the results obtained by Mohan and Raghuraman (1976) for the motion of an assemblage of solid spheres.

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